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ALGEBRAIC EQUATIONS WITH ALMOST-PERIODIC COEFFICIENTS

BY

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INTRODUCTION

It is well known that, generally speaking, continuous one-valued functions of one or more almost-periodic functions lead again to almost-periodic functions. Also it has been shown that in various situations where problems with almost-periodic data give rise to many-valued solutions, these solutions are themselves almost-periodic, or at any rate show interesting almost-periodic features. The case of algebraic functions was first treated by WALTHER[1], who proved that the solutions of an algebraic equation whose coefficients are complex almost-periodic functions of a real variable t are always almost-periodic functions, provided the discriminant $D(t)$ of the equation not merely is different from zero for all t but actually has a positive number as the greatest lower bound of its absolute value. By means of well-established relations between the translation numbers and the Fourier exponents of an almost-periodic function one can easily obtain from WALTHER's proof some first results on the connection between the Fourier exponents of the solutions of the equation and the Fourier exponents of the coefficients.

The latter result was sharpened by CAMERON[2] in an interesting paper dealing with the general question of implicitly given almost-periodic functions. CAMERON also answered in

1) A. WALTHER, Algebraische Funktionen von fastperiodischen Funktionen. Monatshefte für Mathematik und Physik. Bd. 40, 1933, p. 444—457.

2) R. H. CAMERON, Implicit Functions of Almost-periodic Functions. Bulletin of the American Mathematical Society. 1934, p. 895—904.

the negative the question left open by Walther, whether the restriction on the discriminant that $GLB\,|D(t)| > 0$ could be replaced by the weaker condition $D(t) \neq 0$. In fact he stated that there exists an almost-periodic function $f(t)$ which, while $f(t) \neq 0$, has $GLB\,|f(t)| = 0$, and which has the property that the two continuous roots $\pm\sqrt{f(t)}$ of the equation $y^2 - f(t) = 0$ (with discriminant $4f(t)$) are not almost-periodic[1].

In the present paper we investigate systematically the problem of the almost-periodic solutions of an algebraic equation whose coefficients are almost-periodic functions of a real variable and whose discriminant is not near zero. We shall not make use of the previous investigations quoted above, but start afresh; the former results will naturally present themselves in the course of our investigation. The problem will be studied both from an analytical and from an algebraic point of view. That the latter to some degree predominates is simply due to the fact that a certain Abelian substitution group, the "almost-translation group" of the roots of the equation, presents itself as a natural basis for any thorough-going discussion of the problem and turns out to have a fundamental influence on the structure of the solutions. A principal result of our paper is the fact that any arbitrarily given transitive Abelian substitution group can occur as the almost-translation group of the roots of an algebraic equation with almost-periodic coefficients. Thus the solutions of an algebraic equation present a much more rich and varied aspect when the coefficients are general al-

[1] Since Cameron did not indicate explicitly the construction of such a function, the present authors have constructed an example which presumably follows the same lines.

most-periodic functions than in the classical case when the coefficients are pure periodic functions with a common period, which leads to only cyclic groups.

The paper is divided into seven sections. In section I we set down, in such form as will be most convenient for our later applications, some familiar facts concerning Abelian substitution groups and their character groups. In section II we introduce the notion of almost-translation substitution and almost-translation group, basing our considerations on an arbitrary finite set of complex functions. In section III we apply the notions and results of section II to such sets of functions when they form the roots of an algebraic equation with almost-periodic coefficients. Section IV (like section I) is of auxiliary character, and assembles some well known important relations between the translation numbers and the Fourier exponents of one or more almost-periodic functions. In section V we apply these relations to our present problem. The principal contribution of the paper is found in section VI, where we deduce necessary and sufficient conditions that a given finite set of almost-periodic functions shall have as its almost-translation group an arbitrarily given transitive Abelian substitution group. In these conditions the characters of the group play a predominant role; and by means of the characters we are enabled to give a certain canonical representation of the functions considered. We also give general examples — which from various aspects may be said to be the simplest, as well — of sets of functions with arbitrary transitive almost-translation groups. The paper is concluded in section VII by the application of the results of section VI to the original algebraic problem.

I. Preliminary Remarks on Abelian Substitution Groups.

We shall be concerned throughout with substitutions which permute the m objects in a finite set. If the objects are distinguished by attaching the indices $1, \ldots, m$ to a fixed symbol, we shall denote the set by enclosing the fixed symbol in [ ]'s: e. g. $[a]$ denotes the set composed of $a_1, \ldots, a_m$. Where no ambiguity arises we shall not distinguish between a substitution operating on the elements of $[a]$, and the corresponding substitution operating on the indices. Thus we shall customarily denote the substitution S which takes a_1 into $a_{g_1}, \ldots, a_m$ into a_{g_m}, by $\begin{pmatrix} 1, \ldots, m \\ g_1, \ldots, g_m \end{pmatrix}$. We shall indicate the effect of S on any a_h by writing $a_{g_h} = Sa_h$. The substitution resulting from operating first with S_1 and then with S_2 will be denoted by $S_2 S_1$. Finally, a relation which holds between each a_h and its corresponding Sa_h will frequently be denoted by enclosing a specimen relation in [ ]'s. Thus if, as often in the later sections, the a's represent numbers such that $|a_h - Sa_h| \leq \varepsilon$ for $h = 1, \ldots, m$, we shall write simply $[|a_h - Sa_h| \leq \varepsilon]$.

Let Γ be an Abelian group of substitutions on $[a]$ (or on the indices of $[a]$). A generating system of Γ is a set of elements of Γ, say $S'_1, \ldots, S'_q$, in terms of which every element S of Γ can be expressed as a power product, as $S = (S'_1)^{e_1} \ldots (S'_q)^{e_q}$. (If this representation is unique in the sense that each factor $(S'_i)^{e_i}$ is uniquely determined by S, the generating system is called a basis of Γ). A generating system which contains the smallest possible number, say μ, of elements S is called a minimal generating system. This number μ, which is characteristic of the group, has a property which is particularly important for our purposes, namely that out of every generating system $S'_1, \ldots, S'_q$ there can always be chosen μ elements, say $S'_1, \ldots, S'_\mu$, such that (E denoting the identity element) for every relation $(S'_1)^{e_1} \ldots (S'_\mu)^{e_\mu} = E$ between these elements, we have $GCD\,(e_1, \ldots, e_\mu) > 1$.

Of special importance for our investigation is the case where the Abelian substitution group Γ is transitive. Then Γ is of order m, and each substitution of Γ is uniquely determined by specifying its effect on any one element of $[a]$. Hence we can always assign the indices $1, \ldots, m$ to the objects a and the substitutions S in such a way that $S_h a_1 = a_h$, so that $S_1 = \begin{pmatrix} 1, \ldots \\ 1, \ldots \end{pmatrix}, \ldots, S_m = \begin{pmatrix} 1, \ldots \\ m, \ldots \end{pmatrix}$. Such a concordant indexing of $[a]$ and Γ has the readily verifiable advantage that the effect of multiplying every element of Γ by any fixed element S_h is to perform on the elements of Γ a substitution whose symbol (in terms of the indices) is identical with that of S_h. Thus, if $S_h S_1 = S_{h_1}, \ldots, S_h S_m = S_{h_m}$, then $S_{h_1}, \ldots, S_{h_m}$ is a permutation of $S_1, \ldots, S_m$, and $\begin{pmatrix} 1, \ldots, m \\ h_1, \ldots, h_m \end{pmatrix}$, interpreted as a substitution on $[a]$, is precisely S_h.

Since Γ is Abelian we know that a complete set of characters of Γ forms (with respect to ordinary multiplication) a group Γ' isomorphic with Γ. We may then denote the elements of the character group by $\chi_1(S), \ldots, \chi_m(S)$. (We shall denote the principal character by its value, 1).

The notions of generating system and minimal generating system apply to Γ^* as they did to Γ, but with certain additional features due to the connection between the two groups. Thus, we shall find useful for our later applications the following well known simple criterion for a generating system of Γ^*: the characters $\chi_1', \ldots, \chi_q'$ form a generating system of Γ^* if and only if $\chi_1'(S_h) = \chi_1'(S_g), \ldots, \chi_q'(S_h) = \chi_q'(S_g)$ implies $S_h = S_g$. Also, since Γ^* is isomorphic with Γ, a minimal generating system of Γ^* has the same number μ of elements as a minimal generating system of Γ. Further, out of any generating system $\chi_1', \ldots, \chi_q'$ can be chosen μ characters, say $\chi_1', \ldots, \chi_\mu'$, such that every relation $(\chi_1')^{e_1} \ldots (\chi_\mu')^{e_\mu} = 1$ implies that $GCD\ (e_1, \ldots, e_\mu) > 1$. Finally, since Γ is transitive, and since by definition $\chi(S_h)\,\chi(S_g) = \chi(S_h S_g)$, it follows from the previously noted result of indexing the elements of Γ and $[a]$ concordantly: If $\chi(S)$ is any character and $S_{h_1}, \ldots, S_{h_m}$ is just that permutation of $S_1, \ldots, S_m$ produced by applying to the indices of the S's the substitution which denotes S_h, then

$$\chi(S_h)\,\chi(S_1) = \chi(S_{h_1}), \ldots, \chi(S_h)\,\chi(S_m) = \chi(S_{h_m}).$$

We conclude this section with some remarks on those powers of the substitutions of a (not necessarily transitive) group Γ which leave a given element of $[a]$ unaltered. For fixed element a_h we may regard the relation $S^e a_h = a_h$ as an equation in the variable substitution S, whose range is the group Γ. We denote by ν_h the least positive integer e for which this relation is an identity in S. Trivially, ν_h is not greater than the order g of the group, since then $S^g a_h = E a_h = a_h$, not only for every S in Γ, but for every a_h in $[a]$. Equally trivially, $\nu_h \leq m!$, since the set of all substitutions on $[a]$ forms the symmetric group Σ_m of order $m!$. If Γ is Abelian and transitive, and hence of order m, we have the sharper result that ν_h is $\leqq m$; and in fact ν_h is independent of h and equal to m', the least common multiple of the orders of the elements of Γ. If the Abelian group Γ is intransitive, each a_h belongs to a transitivity set containing, say, m_h elements; and Γ is homomorphic with a transitive Abelian group Γ' operating on this subset and hence of order m_h. In this case $\nu_h = m'_h \leqq m_h$, where m'_h is the least common multiple of the orders of the substitutions in Γ'.

II. Almost-Translation Substitutions and the Almost-Translation Group.

In this section we suppose that the m elements of the finite set $[a]$ upon which substitutions are to be performed, are distinct one-valued complex functions, $f_1(t), \ldots, f_m(t)$, of a real variable, defined for $-\infty < t < +\infty$. In accordance with our previous notation the set is then denoted by $[f(t)]$.

We first assume only that each function is continuous for all values of t.

Definition. For given $\varepsilon > 0$ we shall say of a real number τ that *it ε-performs the substitution S on $[f(t)]$* if

$$[\,|f_h(t+\tau) - Sf_h(t)| \leqq \varepsilon] \text{ for } -\infty < t < +\infty.$$

We shall denote by $\{\tau_{(S)}(\varepsilon)\}$ the set of all real numbers τ each of which ε-performs the fixed substitution S on $[f(t)]$; and by $\{\tau_{[f]}(\varepsilon)\}$ the set of all real numbers τ each of which ε-performs some substitution on $[f(t)]$.

Remark. As the m functions are distinct from each other, it is clear (quite apart from any consideration of the continuity of the functions) that for sufficiently small positive ε, no τ can ε-perform more than one substitution on $[f(t)]$. In other words, there exists $\varepsilon^* > 0$ such that if $0 < \varepsilon \leq \varepsilon^*$ and τ belongs to both $\{\tau_{(S_1)}(\varepsilon)\}$ and $\{\tau_{(S_2)}(\varepsilon)\}$, then $S_1 = S_2$.

Lemma. If $\tau_1 \in \{\tau_{(S_1)}(\varepsilon_1)\}$[1] and $\tau_2 \in \{\tau_{(S_2)}(\varepsilon_2)\}$, then $(\tau_1 + \tau_2) \in \{\tau_{(S_2 S_1)}(\varepsilon_1 + \varepsilon_2)\}$ and $(\tau_1 + \tau_2) \in \{\tau_{(S_1 S_2)}(\varepsilon_1 + \varepsilon_2)\}$.

Proof. By assumption $[|f_h(t+\tau_1) - S_1 f_h(t)| \leq \varepsilon_1]$. Replacing t by $t + \tau_2$,

$$(1) \qquad [|f_h(t+\tau_2+\tau_1) - S_1 f_h(t+\tau_2)| \leq \varepsilon_1].$$

Also, by assumption $[|f_h(t+\tau_2) - S_2 f_h(t)| \leq \varepsilon_2]$, whence, replacing f_h by $S_1 f_h$,

$$(2) \qquad [|S_1 f_h(t+\tau_2) - S_2 S_1 f_h(t)| \leq \varepsilon_2].$$

From (1) and (2) follows

$$[|f_h(t+\tau_2+\tau_1) - S_2 S_1 f_h(t)| \leq \varepsilon_1 + \varepsilon_2],$$

which says that $(\tau_2 + \tau_1) \in \{\tau_{(S_2 S_1)}(\varepsilon_1 + \varepsilon_2)\}$. The other half of the lemma is derived similarly (or simply by interchanging the indices 1 and 2).

Corollary. If $\tau \in \{\tau_{(S)}(\varepsilon)\}$ and ν is any positive integer, then $\nu\tau \in \{\tau_{(S^\nu)}(\nu\varepsilon)\}$, or[2]

$$\nu\{\tau_{(S)}(\varepsilon)\} \subseteq \{\tau_{(S^\nu)}(\nu\varepsilon)\}.$$

[1] The symbol $\in$ denotes the relation "belonging to", while the symbol ε is used for positive numbers.

[2] For fixed real ν we shall denote by $\nu\{\tau\}$ the set of all numbers $\nu\tau$.

Definition. *If S is any substitution on $[f(t)]$ such that $\{\tau_{(S)}(\varepsilon)\}$ is relatively dense for every positive ε*, we shall say *that S is an almost-translation substitution on $[f(t)]$*, or that $f(t)]$ admits S as an almost-translation substitution.

Theorem 1. *The set Γ of all almost-translation substitutions admitted by $[f(t)]$ is either vacuous or an Abelian substitution group* (which we shall call the *almost-translation group* of $[f(t)]$).

Proof. If Γ is not empty, let S_1 and S_2 be any substitutions in Γ. For any $\varepsilon > 0$ take arbitrary $\tau_1 \in \left\{\tau_{(S_1)}\left(\frac{\varepsilon}{2}\right)\right\}$ and $\tau_2 \in \left\{\tau_{(S_2)}\left(\frac{\varepsilon}{2}\right)\right\}$. If we put $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ in the lemma above, we see that $\tau_1 + \tau_2$ belongs to both $\{\tau_{(S_2S_1)}(\varepsilon)\}$ and $\{\tau_{(S_1S_2)}(\varepsilon)\}$. But these relations hold for every positive ε and corresponding τ_1, τ_2; in particular, when $\varepsilon \leqq \varepsilon^*$, which requires that $S_2S_1 = S_1S_2$. Thus the product of any two substitutions in Γ is commutative. Furthermore, since the sets $\left\{\tau_{(S_1)}\left(\frac{\varepsilon}{2}\right)\right\}$ and $\left\{\tau_{(S_2)}\left(\frac{\varepsilon}{2}\right)\right\}$ are relatively dense for every $\varepsilon > 0$ (actually we only need to have one set relatively dense and the other not empty), the set of all sums $\tau_1 + \tau_2$, and a fortiori $\{\tau_{(S_2S_1)}(\varepsilon)\}$, is relatively dense for all positive ε's. Thus $S_2S_1 = S_1S_2$ belongs to Γ (which is a subset of the finite group Σ_m), so Γ is an Abelian group.

Corollary. *If $[f(t)]$ has an almost-translation group Γ, then each function $f_h(t)$ is almost-periodic.*

Proof. The identity-substitution E must be in Γ; i. e. every positive ε determines a relatively dense set $\{\tau_{(E)}(\varepsilon)\}$ of real numbers τ for each of which

$$[\,|f_h(t+\tau) - Ef_h(t)| \leqq \varepsilon\,].$$

Since $Ef_h = f_h$, this simply says that for each fixed h, every $\tau_{(E)}(\varepsilon)$ is a translation-number $\tau_{f_h}(\varepsilon)$ of $f_h(t)$[1].

Theorem 2. *A necessary and sufficient condition that $[f(t)]$ admit an almost-translation substitution (and therefore have an almost-translation group Γ) is that the set $\{\tau_{[f]}(\varepsilon)\}$ be relatively dense for each $\varepsilon > 0$.*

Proof. The necessity is immediate. To prove the sufficiency we show that the identity E is actually admitted by $[f(t)]$ as an almost-translation substitution. By assumption the set $\left\{\tau_{[f]}\left(\frac{\varepsilon}{m!}\right)\right\}$ is relatively dense, and to each of the numbers $\tau = \tau_{[f]}\left(\frac{\varepsilon}{m!}\right)$ there corresponds a substitution S (depending on τ) such that $\tau\,\varepsilon\left\{\tau_{(S)}\left(\frac{\varepsilon}{m!}\right)\right\}$. Hence $m!\,\tau$ belongs to $\{\tau_{(S^{m!})}(\varepsilon)\}$, i. e. to $\{\tau_{(E)}(\varepsilon)\}$. Thus the set $\{\tau_{(E)}(\varepsilon)\}$ contains the relatively dense set $m!\left\{\tau_{[f]}\left(\frac{\varepsilon}{m!}\right)\right\}$, and is itself relatively dense.

For the remainder of this section we suppose that we have to do with a finite set $[f(t)]$ of functions, each of which is almost-periodic. By the above corollary, this condition is automatically fulfilled when the set has an almost-translation group. We collect here some remarks concerning this case which will be useful in later sections;

[1] In accordance with the usage prevailing in the literature we shall, throughout the paper, denote by $\tau_f(\varepsilon)$ a translation-number (corresponding to ε) of a single function $f(t)$. We have been careful to differentiate from this the two other symbols of similar type introduced here, viz. $\tau_{(S)}(\varepsilon)$ and $\tau_{[f]}(\varepsilon)$, each of which denotes a number performing an office similar to that of the translation-number of a single function, but in connection with a finite set of functions.

in particular we shall see that the converse of the above corollary is also true, that is that the almost-periodicity of the separate functions $f_h(t)$ implies the existence of an almost-translation group of the set $[f(t)]$.

1°. A familiar and important property of such a set is the fact that each $\varepsilon > 0$ determines a relatively dense set of real numbers τ, each of which is a translation number for every one of the functions $f_h(t)$, so that for $1 \leq h \leq m$,

$$|f_h(t+\tau) - f_h(t)| \leq \varepsilon \text{ for } -\infty < t < +\infty.$$

Even more, as BOCHNER has shown, there exists an almost-periodic function $F(t)$, called a m a j o r a n t of the set, such that for every positive ε the set $\{\tau_F(\varepsilon)\}$ of its translation-numbers is identical with the set-theoretical product of the sets $\{\tau_{f_1}(\varepsilon)\}, \ldots, \{\tau_{f_m}(\varepsilon)\}$ of translation-numbers of the individual functions.

2°. Our set $[f(t)]$ certainly admits the identity E as an almost-translation substitution, and hence has an a l m o s t - t r a n s l a t i o n g r o u p Γ (which may consist of E only). In fact, for every $\varepsilon > 0$, the set $\{\tau_F(\varepsilon)\}$ determined in 1° is precisely the set $\{\tau_{(E)}(\varepsilon)\}$.

3°. If S is such a substitution of the whole group Σ_m that for some $\varepsilon > 0$ the set $\{\tau_{(S)}(\varepsilon)\}$ is n o t e m p t y, then the set $\{\tau_{(S)}(2\varepsilon)\}$ is r e l a t i v e l y d e n s e. For if τ_0 is some number in $\{\tau_{(S)}(\varepsilon)\}$ and $\tau \,\varepsilon\, \{\tau_{(E)}(\varepsilon)\}$, then $(\tau_0 + \tau) \,\varepsilon\, \{\tau_{(S)}(2\varepsilon)\}$, and this set is therefore relatively dense since $\{\tau_{(E)}(\varepsilon)\}$ is relatively dense.

4°. There exists a f i x e d n u m b e r $\varepsilon^{**} > 0$ with the property *that any substitution S of Σ_m belongs to Γ provided only that for some positive $\varepsilon \leq \varepsilon^{**}$, the set $\{\tau_{(S)}(\varepsilon)\}$ is not empty*. For let us denote by $\Delta(\varepsilon)$ the (possibly vacuous, certainly finite) set of substitutions belonging to Σ_m but not to Γ, each of which is ε-performed by some real τ. It is evident that $0 < \varepsilon_1 < \varepsilon_2$ implies $\Delta(\varepsilon_1) \subseteqq \Delta(\varepsilon_2)$, and that this, combined with the finiteness of every $\Delta(\varepsilon)$, implies that for some positive ε^{**}, $0 < \varepsilon_1 \leqq \varepsilon^{**} < \varepsilon_2$ gives us $\Delta(\varepsilon_1) = \Delta(\varepsilon^{**}) \subseteqq \Delta(\varepsilon_2)$. If $\Delta(\varepsilon^{**})$ actually contained a substitution S, for this S every $\{\tau_{(S)}(\varepsilon)\}$ would be non-vacuous, and by 3° every $\{\tau_{(S)}(2\varepsilon)\}$ would be relatively dense. S would then by definition belong to Γ, contrary to hypothesis. Thus, if for some positive $\varepsilon \leqq \varepsilon^{**}$, a substitution S of Σ_m is ε-performed by even one τ, then that substitution belongs to Γ.

5°. Let $f_h(t)$ be any fixed member of $[f(t)]$, and ν_h the corresponding positive integer (defined in section I) for which $S^{\nu_h} f_h(t) = f_h(t)$ for every S in Γ. Then for each sufficiently small ε (in fact for $\varepsilon \leqq \nu_h \varepsilon^{**}$),

$$\left\{\tau_{f_h}(\varepsilon)\right\} \supseteqq \nu_h \left\{\tau_{[f]}\left(\frac{\varepsilon}{\nu_h}\right)\right\}.$$

For, for each S in Γ, $\left\{\tau_{f_h}(\varepsilon)\right\} \supseteqq \left\{\tau_{(S^{\nu_h})}(\varepsilon)\right\}$, and $\left\{\tau_{(S^{\nu_h})}(\varepsilon)\right\} \supseteqq \nu_h \left\{\tau_{(S)}\left(\frac{\varepsilon}{\nu_h}\right)\right\}$, and hence

$$\left\{\tau_{f_h}(\varepsilon)\right\} \supseteqq \nu_h \left\{\tau_{(S)}\left(\frac{\varepsilon}{\nu_h}\right)\right\}.$$

But the last relation, just proved for each S in Γ, holds for trivial reasons for every S in Σ_m which does not belong to Γ, since for such an S the set $\left\{\tau_{(S)}\left(\frac{\varepsilon}{\nu_h}\right)\right\}$ is empty on account of 4° $\left(\text{as } \frac{\varepsilon}{\nu_h} \leqq \varepsilon^{**}\right)$. Now the set-theoretical sum, taken over all the S's in Σ_m, of the sets $\left\{\tau_{(S)}\left(\frac{\varepsilon}{\nu_h}\right)\right\}$, is exactly the set $\left\{\tau_{[f]}\left(\frac{\varepsilon}{\nu_h}\right)\right\}$; so $\left\{\tau_{f_h}(\varepsilon)\right\} \supseteqq \nu_h \left\{\tau_{[f]}\left(\frac{\varepsilon}{\nu_h}\right)\right\}$, and the proof is completed.

III. The Existence of an Almost-Translation Group for the Solutions of an Algebraic Equation with Almost-Periodic Coefficients.

Let

$$(3) \qquad y^m + x_1(t)\, y^{m-1} + \ldots + x_{m-1}(t)\, y + x_m(t) = 0$$

be an algebraic equation of degree m in the complex variable y, with leading coefficient 1 and remaining coefficients, $x_1(t), \ldots, x_m(t)$, almost-periodic functions of the real variable t. We denote the discriminant of the equation (which, by the way, is also an almost-periodic function) by

$$D(t) = d\left[x_1(t), \ldots, x_m(t)\right].$$

If $D(t) \neq 0$ for $-\infty < t < +\infty$, the m roots of the equation are distinct for every value of t, and since the coefficients are continuous, these roots may be sorted in just one way (except for choice of notation) into m one-valued functions, $y_1(t), \ldots, y_m(t)$, each of which is continuous for all values of t. Further, as the coefficients $x_j(t)$ are bounded and uniformly continuous, the roots $y_h(t)$ will also be bounded and uniformly continuous in $-\infty < t < +\infty$.

We now assume not merely that $D(t) \neq 0$, but that

$$|D(t)| > \alpha > 0.$$

Then there clearly exists a $\beta > 0$ such that for every t

$$(4) \qquad |y_h(t) - y_g(t)| > \beta \text{ for } h \neq g.$$

Consequently, for any two values t_1 and t_2, there can exist at most one substitution S (we may denote it by $S(t_1, t_2)$ to indicate its dependence on t_1 and t_2) such that

$$\left[\, |y_h(t_1) - Sy_h(t_2)| \leq \frac{\beta}{2} \,\right].$$

We shall now prove the important theorem:

Theorem 3. *The set* $\left[y_h(t)\right]$ *of continuous solutions of an equation* (3) *with almost-periodic coefficients* $x_j(t)$ *and discriminant satisfying* $|D(t)| > \alpha > 0$, *has an almost-translation group* Γ, *and hence is composed of almost-periodic functions.*

The last fact was first given by Walther, as was mentioned in the introduction.

Proof: According to theorem 2 of section II it suffices to show that to every given $\varepsilon > 0$ there corresponds a relatively dense set of real numbers $\left\{\tau_{[y]}(\varepsilon)\right\}$, i. e. of numbers τ to each of which there corresponds some substitution $S = S(\tau)$ of Σ_m such that $[|y_h(t+\tau) - Sy_h(t)| \leqq \varepsilon]$. Naturally it suffices to consider "small" positive ε's; we may therefore take the given ε to be $< \frac{\beta}{2}$, where β is the positive number occurring in the inequality (4). With this restriction on ε, if for some two values t_1 and t_2 we have found two substitutions S_1 and S_2 such that

$$[|y_h(t_1) - S_1 y_h(t_2)| \leqq \varepsilon] \text{ and } [|y_h(t_1) - S_2 y_h(t_2)| \leqq \varepsilon],$$

we may conclude that $S_1 = S_2$.

Corresponding to the given ε we determine (as is possible because the coefficients $x_j(t)$ are bounded) a $\delta > 0$ such that for any two real numbers t' and t'' satisfying

$$|x_j(t') - x_j(t'')| \leqq \delta \quad (j = 1, \ldots, m)$$

there is a substitution $S = S(t', t'')$, for which

$$\left[|y_h(t') - Sy_h(t'')| \leqq \frac{\varepsilon}{3}\right].$$

From the preceding remark this substitution is uniquely determined since $\frac{\varepsilon}{3} < \varepsilon < \frac{\beta}{2}$.

Consider the relatively dense set $\left\{\tau_X(\delta)\right\}$ of translation numbers (corresponding to this δ) of a majorant $X(t)$ of the coefficients $x_j(t)$. We shall prove the relative density of $\left\{\tau_{[y]}(\varepsilon)\right\}$ by showing that $\left\{\tau_{[y]}(\varepsilon)\right\} \supseteq \left\{\tau_X(\delta)\right\}$. In other words, we shall show that to each fixed τ satisfying for all t the inequalities

$$|x_j(t+\tau)-x_j(t)| \leq \delta \quad (j = 1, \ldots, m)$$

there corresponds a substitution $S = S(\tau)$, such that

$$[|y_h(t+\tau)-Sy_h(t)| \leq \varepsilon] \qquad (-\infty < t < +\infty).$$

Note first that, from the manner of choosing δ, our fixed τ and any fixed t certainly determine one, and only one, substitution $S = S(t,\tau)$ which satisfies the relation

$$\left[|y_h(t+\tau)-Sy_h(t)| \leq \frac{\varepsilon}{3}\right].$$

The proof will obviously be complete when we have shown that this substitution $S(t,\tau)$ is independent of t, i. e. depends only on τ. For this purpose we first determine an $\eta > 0$ such that the inequality $|t'-t''| \leq \eta$ implies the inequalities $|y_h(t')-y_h(t'')| \leq \frac{\varepsilon}{3}$ for $1 \leq h \leq m$, i. e. implies

$$\left[|y_h(t')-Ey_h(t'')| \leq \frac{\varepsilon}{3}\right].$$

Since the whole real axis can be covered by intervals of length η, in order to prove that $S(t,\tau)$ is independent of t it suffices to prove that for two arbitrary numbers t_1 and t_2 satisfying $|t_1-t_2| \leq \eta$, the two substitutions $S_1 = S(t_1,\tau)$, and $S_2 = S(t_2,\tau)$ are identical.

From the definition of S_1 and S_2 we have

$$\left[|y_h(t_1+\tau)-S_1 y_h(t_1)| \leqq \frac{\varepsilon}{3}\right] \text{ and } \left[|y_h(t_2+\tau)-S_2 y_h(t_2)| \leqq \frac{\varepsilon}{3}\right].$$

Since $|(t_1+\tau)-(t_2+\tau)| = |t_1-t_2| \leqq \eta$, we have also

$$\left[|y_h(t_1+\tau)-Ey_h(t_2+\tau)| \leqq \frac{\varepsilon}{3}\right].$$

Hence

$$\left[|S_1 y_h(t_1)-S_2 y_h(t_2)| \leqq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\right].$$

Replacing y_h by $S_1^{-1} y_h$, this becomes

$$[|y_h(t_1)-S_2 S_1^{-1} y_h(t_2)| \leqq \varepsilon]. \tag{5}$$

On the other hand, from $|t_1-t_2| \leqq \eta$ we get

$$\left[|y_h(t_1)-Ey_h(t_2)| \leqq \frac{\varepsilon}{3} < \varepsilon\right]. \tag{6}$$

But (as we emphasized above) relations (5) and (6) enable us to conclude that $S_2 S_1^{-1}$ and E are identical. Thus $S_1 = S_2$, and the proof is completed.

We here introduce an abbreviation of our notation which will be useful later on. If $f(t)$ and $g(t)$ are almost-periodic functions we shall write

$$\{\tau_f\} \subseteq \{\tau_g\}$$

in order to indicate that, corresponding to any $\varepsilon_1 > 0$ (or, equally well, to any sufficiently small $\varepsilon_1 > 0$), there exists an $\varepsilon_2 > 0$ such that $\{\tau_f(\varepsilon_2)\} \subseteq \{\tau_g(\varepsilon_1)\}$. Such symbolic abbreviations we shall use not only in connection with the translation numbers of single almost-periodic functions, but also for translation numbers $\tau_{[f]}(\varepsilon)$ and $\tau_{(S)}(\varepsilon)$ referring to sets of functions. For instance, by

$$\{\tau_{[f]}\} \subseteq \{\tau_g\}$$

we shall mean that to any $\varepsilon_1 > 0$ there corresponds an $\varepsilon_2 > 0$ such that $\{\tau_{[f]}(\varepsilon_2)\} \subseteq \{\tau_g(\varepsilon_1)\}$.

In terms of this notation we see that, for our algebraic equation with almost-periodic coefficients $x_j(t)$ and $|D(t)| > \alpha > 0$, the translation numbers of the majorant $X(t)$ and the translation numbers belonging to the set of roots $[y(t)]$ are in so far the same as

$$\{\tau_X\} \subseteq \{\tau_{[y]}\} \quad \text{and} \quad \{\tau_{[y]}\} \subseteq \{\tau_X\}.$$

The first of these two relations has been directly shown in the course of the proof above. The second relation holds because, corresponding to any given $\varepsilon_1 > 0$, since the roots $y_h(t)$ are bounded, we can find a positive ε_2 so small that every τ which ε_2-performs some substitution on $[y(t)]$ must satisfy, for all t, the inequalities

$$|x_j(t+\tau) - x_j(t)| \leq \varepsilon_1 \qquad (j = 1, \ldots, m).$$

IV. Auxiliary Remarks on Relations between Translation Numbers and Fourier Exponents of Almost-Periodic Functions.

With any almost-periodic function $f(t)$ there is associated a certain series as its Fourier series,

$$f(t) \sim \sum a_n e^{i\lambda_n t}.$$

In the treatment of various questions concerning the Fourier exponents λ_n of $f(t)$, it is not just the set of exponents themselves which is of primary importance, but the larger set containing all linear combinations with integral coefficients of a finite number of these exponents,

$$g_1\lambda_1 + g_2\lambda_2 + \ldots + g_N\lambda_N.$$

This set is usually denoted by M_f and is called the module of the function $f(t)$; evidently it is the smallest number-module which contains all the exponents λ_n.

There exist important relations between the translation numbers $\tau_f(\varepsilon)$ and the exponents λ_n of an almost-periodic function $f(t)$. Especially we shall have to use a necessary and sufficient condition[1] that the module M_g of an almost-periodic function $g(t)$ be contained in the module M_f of an almost-periodic function $f(t)$, expressed in terms of the translation numbers of the two functions. By using the abbreviated notation introduced at the end of the previous section we can express this condition very simply as follows:

Lemma 1. Let $f(t)$ and $g(t)$ be two almost-periodic functions. In order that $M_g \subseteq M_f$ it is necessary and sufficient that $\{\tau_f\} \subseteq \{\tau_g\}$.

Roughly speaking, the fewer translation numbers, the more exponents.

We shall also recall a relation between the translation numbers and the Fourier exponents of a single almost-periodic function:

Lemma 2. In order to show that the real number θ belongs to the module M_f of an almost-periodic function $f(t) \sim \sum a_n e^{i\lambda_n t}$, it suffices to show that to any positive ε

[1] Compare H. Bohr, Ueber fastperiodische ebene Bewegungen. Commentarii Mathematici Helvetici. Vol. 4, 1932, p. 51—64.

there corresponds a positive $\delta = \delta(\varepsilon)$, such that each τ belonging to $\{\tau_f(\delta)\}$ satisfies the inequality

$$\left| e^{i\theta\tau} - 1 \right| \leqq \varepsilon.$$

We shall not enter upon the proof of these two lemmas, but content ourselves with pointing out that they depend essentially on a famous theorem of KRONECKER on Diophantine approximations. As we shall make direct use of KRONECKER's theorem in section VI, but in a form slightly different from the usual one, we take this occasion to re-state his theorem in this form, using the exponential function e^{it}.

KRONECKER's Theorem. Let $\lambda_1, \ldots, \lambda_N$ be arbitrarily given real numbers, and let $\eta_1, \ldots, \eta_N$ be given complex numbers of absolute value 1. In order that to each positive ε there correspond a real τ satisfying the N inequalities

$$\left| e^{i\lambda_n\tau} - \eta_n \right| \leqq \varepsilon \quad (n = 1, \ldots, N),$$

it is necessary and sufficient that whenever a linear relation with integral coefficients

$$g_1\lambda_1 + \ldots + g_N\lambda_N = 0$$

holds between the λ's, the corresponding relation

$$\eta_1^{g_1} \ldots \eta_N^{g_N} = 1$$

hold between the η's.

One sees immediately that the condition is necessary; the real content of the theorem lies in the fact that it is also sufficient.

V. Relations between the Fourier Exponents of the Roots and the Fourier Exponents of the Coefficients of an Algebraic Equation.

We return to the algebraic equation

$$y^m + x_1(t)\, y^{m-1} + \ldots + x_{m-1}(t)\, y + x_m(t) = 0$$

of section III. We denote by $X(t)$ some majorant of the almost-periodic coefficients $x_1(t), \ldots, x_m(t)$. As is well known, the module M_X is independent of the choice of the majorant, and is not only the smallest number-module containing all the Fourier exponents of $X(t)$, but is also the smallest number-module containing all the Fourier exponents of all the functions $x_j(t)$.

As before, we assume that the discriminant of the equation satisfies the inequality

$$\underset{-\infty < t < \infty}{G\,L\,B} \; |D(t)| > 0,$$

so we know that the equation has as continuous roots a set of almost-periodic functions,

$$y_1(t), \ldots, y_m(t).$$

Thus the roots have a majorant $Y(t)$, and the module M_Y is the smallest number-module containing all the Fourier exponents of the functions in the set $[y(t)]$.

In this section we shall discuss the connection between the Fourier exponents of the roots of the equation and the Fourier exponents of its coefficients. The connection will be exhibited by demonstrating some important relations between the modules M_X, M_Y, and M_{y_h}, and their multiples.[1)]

[1)] By the multiple $r \cdot M$ we shall understand the module arising from the module M by multiplying every number in M by r.

Theorem 4. $M_X \subseteq M_Y$.

Proof. This is an immediate consequence of the facts: that the coefficients $x_j(t)$ are polynomials in the roots $y_h(t)$; and that the Fourier series of a polynomial function of almost-periodic functions is determined simply by formal calculation from the Fourier series of these functions. Thus each exponent of each $x_j(t)$ is a linear combination with integral coefficients of the exponents of the roots $y_1(t), \ldots, y_m(t)$, and therefore belongs to M_Y. Hence the whole module M_X is contained in M_Y.

We now proceed to a theorem which is not at all trivial, and the proof of which uses essentially the connection between Fourier exponents and translation numbers mentioned in section IV. As in section II we denote by Γ the almost-translation group of the set $[y(t)]$ of roots, and by $\nu_h (\leq m)$ the least positive integer e such that $S^e y_h = y_h$ for every S in Γ, as defined in section I.

Theorem 5. *For each h among $1, \ldots, m$ we have the relation*

$$M_{y_h} \subseteq \frac{1}{\nu_h} M_X. \tag{7}$$

The fact that to each of the roots $y_h(t)$ there corresponds some number $\nu_h \leq m$ for which $M_{y_h} \subseteq \dfrac{1}{\nu_h} M_X$, was first found by Cameron in the paper quoted in the introduction.

Proof. In remark 5° of section II it was shown that for each sufficiently small ε,

$$\nu_h \left\{ \tau_{[y]} \left(\frac{\varepsilon}{\nu_h} \right) \right\} \subseteq \left\{ \tau_{y_h}(\varepsilon) \right\}.$$

Hence, using the abbreviation introduced at the end of section III, we have

$$\nu_h \{ \tau_{[y]} \} \subseteq \{ \tau_{y_h} \}. \tag{8}$$

But, as shown in section III, the translation numbers $\tau_{[y]}$ and τ_X are the "same" in the sense that

$$\{ \tau_{[y]} \} \subseteq \{ \tau_X \} \quad \text{and} \quad \{ \tau_X \} \subseteq \{ \tau_{[y]} \}.$$

Hence the relation (8) is equivalent to the relation

$$\nu_h \{ \tau_X \} \subseteq \{ \tau_{y_h} \}.$$

Now it is evidently the same thing to say of a number τ that it is a $\nu_h \cdot \tau_X(\delta)$ as to say that it is a $\tau_{X^*}(\delta)$, where $X^*(t)$ denotes the function $X\left(\frac{t}{\nu_h}\right)$. Thus we may write

$$\{ \tau_{X^*} \} \subseteq \{ \tau_{y_h} \},$$

and by lemma 1 of section IV we conclude that

$$M_{y_h} \subseteq M_{X^*}.$$

But the Fourier exponents of $X^*(t) = X\left(\frac{t}{\nu_h}\right)$ are simply the exponents of $X(t)$ itself, each divided by ν_h. Hence $M_{X^*} = \frac{1}{\nu_h} M_X$, and we get the desired result,

$$M_{y_h} \subseteq \frac{1}{\nu_h} M_X.$$

Corollary 1.

$$M_Y \subseteq \frac{1}{m!} M_X.$$

For this merely contracts to one relation the m relations $M_{y_h} \subseteq \frac{1}{m!} M_X$, each of which must hold since every ν_h, being $\leq m$, is a divisor of $m!$. This corollary, substantially due to WALTHER, may also be proved readily without using the sharper relations (7).

Corollary 2. The modules M_X and M_Y have the same maximum number of rationally independent elements (i. e. elements linearly independent with respect to the rational domain).

This follows immediately from $M_X \subseteqq M_Y$ and $M_Y \subseteqq \frac{1}{m!} M_X$.

In the remainder of the paper we shall confine our attention to the case where the almost-translation group Γ is transitive. In this case we get a rather complete survey of the relations connecting the Fourier series of the different roots. This restriction is a natural one since, when our given algebraic equation has as roots a set of functions with an intransitive group Γ, it may be split up into a number of equations of lower degree, each one having as its roots the functions of a transitivity system of the original set; at the same time, as we proceed to show, this process does not enlarge the modules of the coefficients, i. e. the module of the coefficients of each new equation is contained in the original module M_X.

For let $y_1(t), \ldots, y_\varrho(t)$ be a transitivity system of the set $[y(t)]$, and let

$$\big(y-y_1(t)\big)\ldots\big(y-y_\varrho(t)\big) = y^\varrho + \xi_1(t)\, y^{\varrho-1} + \ldots + \xi_\varrho(t) = 0$$

be the corresponding new equation, with $\Xi(t)$ as a majorant of its coefficients. Then we have to show that

$$M_\Xi \subseteqq M_X.$$

Now, roughly speaking, every "fine" translation number of $X(t)$ gives rise to one of the substitutions in Γ, and hence, when applied to the transitivity system $y_1(t), \ldots, y_\varrho(t)$, merely effects some permutation of this subset. But this means that it is a "fine" translation number of $\Xi(t)$. Speaking precisely, we have

$$\{\tau_X\} \subseteq \{\tau_\Xi\},$$

and therefore

$$M_\Xi \subseteq M_X.$$

VI. The Fourier Series of a finite Set of Almost-Periodic Functions with a Transitive Almost-Translation Group.

In section II we introduced the notion of the almost-translation group of a finite set $[f(t)]$ of m distinct complex functions of t, each function defined and continuous for $-\infty < t < +\infty$. We there showed that a necessary and sufficient condition that such a set of functions have an almost-translation group Γ is that each function be almost-periodic. In section III we found that in order that an algebraic equation of degree m in the complex variable y with leading coefficient 1 have as roots a set $[y(t)]$ of m distinct almost-periodic functions of t (i. e. a set of m functions having an almost-translation group), it is sufficient that (a) the coefficients be almost-periodic functions of t, and (b) the discriminant $D(t)$ satisfy the inequality

$$\underset{-\infty < t < \infty}{G\,L\,B} \, |D(t)| > 0. \tag{9}$$

Condition (a) is evidently necessary as well; for the coefficients of the equation, being symmetric polymials in the roots, must be almost-periodic themselves. As regards the condition (b), concerning the discriminant, the situation is more complicated. In fact on the one hand, as mentioned in the introduction, condition (b) cannot be replaced by the weaker condition, $D(t) \neq 0$ for all t; on the other hand there exist algebraic equations of degree m with almost-periodic coefficients and satisfied by m distinct almost-periodic functions, for which not only $G\,L\,B\,|D(t)| = 0$ but even $D(t) = 0$ for some t[1]. Thus the restriction (9) on the discriminant introduces an element extraneous to the notion of the almost-translation group. In the present section, therefore, where we deal with those properties stemming directly from the group, we shall not think of our functions as roots of an algebraic equation, but only as having an almost-translation group Γ. As pointed out at the close of the previous section, we consider only the case where Γ is transitive.

Let, then, $[f(t)]$ be a finite set of m distinct almost-periodic functions having as almost-translation group a transitive Abelian group Γ of substitutions on the set. As we remarked in section I, the group has m elements which we may (and do) index concordantly with the functions, so that $S_h f_1 = f_h$. We begin by deducing a number of properties of the Fourier series of the functions $f_h(t)$.

1°. Every function $f_h(t)$, $h = 2, \ldots, m$, has exactly the

1) This may occur even in the case where Γ is transitive. A simple example is given by the two (periodic) functions

$$y_1(t) = \frac{1}{2}(e^{it} + e^{-it}) = \cos t, \quad y_2(t) = -\frac{1}{2}(e^{it} + e^{-it}) = -\cos t$$

with the translation group $\Gamma = \Sigma_2$ of order 2, where the discriminant $D(t) = 4\cos^2 t$ of the corresponding equation

$$(y - y_1(t)) \cdot (y - y_2(t)) = y^2 - \cos^2 t = 0$$

vanishes at $t = \frac{\pi}{2} + n\pi$.

same Fourier exponents, $\lambda_1, \lambda_2, \ldots$, as $f_1(t)$; and the Fourier coefficients in $f_h(t)$ and $f_1(t)$ belonging to the same exponent λ_n, have the same numerical value. Thus we may write the Fourier series of the functions in the form

$$f_h(t) \sim \sum \eta_{h,n} a_n e^{i\lambda_n t} \quad (h = 1, \ldots, m),$$

where each factor $\eta_{h,n}$ has the numerical value 1. Within the restriction that their numerical values be 1, the factors $\eta_{h,n}$ in the Fourier series of any one function may of course be selected quite arbitrarily. Thus, while keeping the notation above for convenience, we shall suppose that each $\eta_{1,n}$ in the expansion of $f_1(t)$ is equal to 1.

Proof. Corresponding to any h among $2, \ldots, m$, and to an arbitrary sequence of positive numbers ε_ν tending to zero, we can choose a sequence of translation numbers $\tau_\nu \,\varepsilon \left\{\tau_{(S_h)}(\varepsilon_\nu)\right\}$. Then the sequence of functions $f_1(t+\tau_\nu)$ will tend to $f_h(t)$ uniformly throughout $-\infty < t < +\infty$, that is the function $f_h(t)$ belongs to the class of almost-periodic functions usually called the uniform closure of the set $\{f_1(t+k)\}$ and denoted by $C_U\{f_1(t+k)\}$. It is a well-known property of such a set that every function in it has the properties ascribed above to $f_h(t)$.

2°. The factors $\eta_{1,n}, \ldots, \eta_{m,n}$ corresponding to any Fourier exponent λ_n are exactly the respective values, $\chi_n(S_1), \ldots, \chi_n(S_m)$, of some character χ_n of the group Γ.

It will be convenient to represent this assertion by the following scheme, where, fixing upon any λ_n, we have dropped the index n momentarily and have arranged in vertical columns the terms and the factors η corresponding to this exponent in the Fourier series of $f_1(t), \ldots, f_m(t)$.

$a\eta_1 e^{i\lambda t}$	$\eta_1 = \chi(S_1)$
$a\eta_2 e^{i\lambda t}$	$\eta_2 = \chi(S_2)$
$\vdots$	$\vdots$
$a\eta_m e^{i\lambda t}$	$\eta_m = \chi(S_m)$

Proof. From the definition of a character it follows that we have to show that if S_{h_1} and S_{h_2} are any (not necessarily distinct) substitutions in Γ, and if $S_{h_1} S_{h_2} = S_{h_3}$, then

$$\eta_{h_1} \cdot \eta_{h_2} = \eta_{h_3}.$$

Take again an arbitrary sequence of positive numbers converging to 0, say $\varepsilon_\nu \to 0$, and for each ν denote by $\tau_{1\nu}$ and $\tau_{2\nu}$ arbitrarily chosen translation numbers belonging to $\{\tau_{(S_{h_1})}(\varepsilon_\nu)\}$ and $\{\tau_{(S_{h_2})}(\varepsilon_\nu)\}$ respectively. Then by the lemma of section II the numbers $\tau_{3\nu} = \tau_{1\nu} + \tau_{2\nu}$ belong to $\{\tau_{(S_{h_3})}(2\varepsilon_\nu)\}$. Hence, as $\nu \to \infty$, we have, uniformly throughout $-\infty < t < +\infty$,

$$f_1(t+\tau_{1\nu}) \to f_{h_1}(t), \quad f_1(t+\tau_{2\nu}) \to f_{h_2}(t), \quad f_1(t+\tau_{3\nu}) \to f_{h_3}(t).$$

It is a trivial fact concerning almost-periodic functions that uniform convergence of a sequence of such functions implies ordinary convergence of the coefficients belonging to any exponent. Since the coefficient corresponding to the exponent λ in the Fourier series of $f_1(t+\tau)$, for arbitrary τ, is given by $a e^{i\lambda\tau}$, and since $a \neq 0$, we conclude from the uniformly convergent sequences above that

$$e^{i\lambda\tau_{1\nu}} \to \eta_{h_1}, \quad e^{i\lambda\tau_{2\nu}} \to \eta_{h_2}, \quad e^{i\lambda\tau_{3\nu}} \to \eta_{h_3}.$$

Finally, since $e^{i\lambda\tau_{1\nu}} \cdot e^{i\lambda\tau_{2\nu}} = e^{i\lambda\tau_{3\nu}}$ for every ν, we have in the limit $\eta_{h_1} \cdot \eta_{h_2} = \eta_{h_3}$.

3°. Those characters of Γ whose values actually occur (in accordance with property 2°) as coefficients in the Fourier series of $f_1(t), \ldots, f_m(t)$, form a generating system of the character group Γ^*.

Proof. Let us denote by $\chi_1', \chi_2', \ldots, \chi_q'$ the distinct characters in Γ^* which thus appear. On the analogy of the scheme in 2°, we write their values in vertical columns, thus:

$$\boxed{\begin{matrix}\chi_1'(S_1)\\ \chi_1'(S_2)\\ \vdots\\ \chi_1'(S_m)\end{matrix}}\,,\quad \boxed{\begin{matrix}\chi_2'(S_1)\\ \chi_2'(S_2)\\ \vdots\\ \chi_2'(S_m)\end{matrix}}\,,\;\ldots,\;\boxed{\begin{matrix}\chi_q'(S_1)\\ \chi_q'(S_2)\\ \vdots\\ \chi_q'(S_m)\end{matrix}}$$

If we now apply the criterion for a generating system of Γ^* which we gave in section I, we see that the denial of our assertion is equivalent to asserting that for some $h \neq g$ the values in row h are identical with the corresponding values in row g. But this would make the Fourier series of $f_h(t)$ and $f_g(t)$ identical, which (from the uniqueness theorem in the theory of almost-periodic functions) would imply $f_h(t) = f_g(t)$, contrary to hypothesis.

4°. If any linear relation with integral coefficients, say

$$g_1\lambda_1 + \ldots + g_N\lambda_N = 0, \tag{10}$$

connects a finite number of the Fourier exponents λ_n, then we have the multiplicative relation

$$\chi_1^{g_1} \ldots \chi_N^{g_N} = 1 \tag{11}$$

connecting the characters $\chi_1, \ldots, \chi_N$ corresponding (as in property 2°) to the exponents $\lambda_1, \ldots, \lambda_N$, respectively.

The relation (11) may be expressed in terms of the values of the characters as a set of m relations

$$(12) \qquad [\chi_1(S_h)]^{g_1} \ldots [\chi_N(S_h)]^{g_N} = 1 \qquad (1 \leqq h \leqq m).$$

Proof. Instead of proving the assertion for some arbitrarily chosen relation (10) it will be more convenient to prove it simultaneously for all linear relations with last index equal to an arbitrarily chosen fixed N. As before, for fixed h, arbitrary sequence of positive numbers $\varepsilon_\nu \to 0$, and corresponding sequence τ_ν of translation numbers with each $\tau_\nu \varepsilon \{\tau_{(S_h)}(\varepsilon_\nu)\}$, we have, uniformly throughout $-\infty < t < +\infty$, that $f_1(t+\tau_\nu) \to f_h(t)$, and hence for each n $(= 1, 2, \ldots)$

$$e^{i\lambda_n \tau_\nu} \to \eta_{h,n} = \chi_n(S_h).$$

Hence for any arbitrarily small ε the N inequalities

$$|e^{i\lambda_n \tau} - \chi_n(S_h)| \leqq \varepsilon \; (n = 1, \ldots, N)$$

are satisfied by some τ (in fact by every τ_ν with sufficiently great index ν). According to the "trivial" part of KRONECKER's theorem stated in the end of section IV, this requires that, corresponding to each relation (10) with last index N, there must hold the corresponding relation (12). Thus (11) holds.

We now turn the problem about, i. e. start from a given transitive Abelian substitution group Γ, and ask what conditions a set of almost-periodic functions $f_h(t)$ must fulfill in order that they be distinct and that $[f(t)]$

have Γ as its almost-translation group. To avoid misunderstanding, however, we emphasize that so far we do not know at all whether every type of transitive Abelian substitution group can occur as the almost-translation group of some set $[f(t)]$. Only after having proved the theorem below shall we take up the main problem of determining whether the conditions stated in the theorem can really be fulfilled for any given group of this sort.

Theorem 6. *Let $[f(t)]$ be a finite set of m almost-periodic functions, $f_1(t), \ldots, f_m(t)$; and let Γ be an arbitrary transitive Abelian group of m substitutions which we denote by $S_1 = \begin{pmatrix} 1, \ldots \\ 1, \ldots \end{pmatrix}, \ldots, S_m = \begin{pmatrix} 1, \ldots \\ m, \ldots \end{pmatrix}$. Then in order that $[f(t)]$ be composed of distinct functions and have Γ as its almost-translation group, it is not only necessary* (as was shown above) *but also* s u f f i c i e n t *that the following four conditions be fulfilled:*

1. *All the functions $f_h(t)$ have exactly the same Fourier exponents λ_n, and the absolute values of the corresponding Fourier coefficients are the same.*

2. *Further, the Fourier series of the functions $f_h(t)$ have the form*

$$f_h(t) \sim \sum \chi_n(S_h) a_n e^{i\lambda_n t}, \tag{13}$$

where $\chi_n(S)$, for each n, is a character of the group Γ.

3. *Those characters $\chi_n(S)$ which actually occur in* (13) *form a generating system of the character-group Γ^*.*

4. *If any finite set, $\lambda_1, \ldots, \lambda_N$, of the Fourier exponents are connected by a linear relation*

$$g_1\lambda_1 + \ldots + g_N\lambda_N = 0$$

with integral coefficients, then the corresponding characters are connected by the relation

$$\chi_1^{g_1} \ldots \chi_N^{g_N} = 1\,.$$

Proof. That m almost-periodic functions whose Fourier series have the form (13) must be distinct, is immediately to be seen from condition 3, which implies that for no two distinct indices, h and g, are the Fourier series of $f_h(t)$ and $f_g(t)$ formally the same. (Here we use only the trivial fact that two distinct Fourier series cannot belong to the same function, in contrast to the corresponding point in the necessity proof, where we had to use the uniqueness theorem, that two distinct functions cannot have the same Fourier series.) Since our functions are distinct, they have an almost-translation group Γ'; and the main point of the proof is to show that condititions 1—4 insure that $\Gamma' = \Gamma$.

We first observe that it suffices to show that each of the substitutions of Γ belongs to Γ'. In fact, when this has been shown, it follows immediately that Γ' must be identical with Γ. For then Γ' must be transitive (since it contains the transitive sub-group Γ) and, being Abelian, cannot contain more than m substitutions.

By way of preparation we make the preliminary remark that the set-theoretical sum $\{g(t)\}$ of the m sets of functions

$$\{f_1(t+k)\}, \ldots, \{f_m(t+k)\} \qquad (-\infty < k < +\infty)$$

is a majorisable set of almost-periodic functions, in fact majorisable by any majorant $F(t)$ of the set $[f(t)]$; and that every function in $\{g(t)\}$ has the same Fourier exponents $\lambda_1, \lambda_2, \ldots$ as each function in $[f(t)]$. Hence, according to a wellknown theorem on majorisable sets, due to Bochner, it holds, roughly speaking, that formal convergence of the Fourier series of a sequence of functions drawn from the set $\{g(t)\}$ (i. e. actual convergence of the coefficients belonging to each fixed exponent) implies uniform convergence of the sequence of functions. Exactly speaking: to each $\varepsilon > 0$ there corresponds a positive integer N and a $\delta > 0$ such that any two functions $g'(t) \sim \sum b'_n e^{i\lambda_n t}$ and $g''(t) \sim \sum b''_n e^{i\lambda_n t}$, of the set $\{g(t)\}$, the coefficients of which fulfill the inequalities

$$|b'_n - b''_n| \leqq \delta \quad (n = 1, \ldots, N),$$

themselves satisfy the inequality

$$|g'(t) - g''(t)| \leqq \varepsilon \quad (-\infty < t < +\infty).$$

Now let S_h be an arbitrarily chosen fixed element of Γ. We shall show that S_h also belongs to Γ', i. e. that S_h is an almost-translation substitution of the set $[f(t)]$. From remark 4° in section II it suffices to prove that to some positive $\varepsilon \leqq \varepsilon^{**}$ there corresponds at least one real number τ which satisfies for all values of t the m inequalities

$$|f_l(t+\tau) - S_h f_l(t)| \leqq \varepsilon \quad (l = 1, \ldots, m). \tag{14}$$

We first make clear, for each l, which of the functions $f_1, \ldots, f_m$ is denoted by $S_h f_l$. This can be seen at once by considering the Fourier series of the function:

$$f_l(t) \sim \sum \chi_n(S_l) a_n e^{i\lambda_n t}.$$

In this Fourier series the index l occurs only as index for that substitution S_l of which the characters χ_n are taken. As we noted in section I, since χ_n is a character of Γ, the substitution S_h performed on the index l simply results in replacing $\chi_n(S_l)$ by $\chi_n(S_h S_l)$, that is by the product $\chi_n(S_h)\chi_n(S_l)$. Hence the function $S_h f_l(t)$ is just that function among $f_1(t), \ldots, f_m(t)$ whose Fourier series is given by

$$\sum \chi_n(S_h)\chi_n(S_l) a_n e^{i\lambda_n t}.$$

We now proceed to prove that there exists a τ which satisfies the inequalities (14). As the Fourier series of $f_l(t+\tau)$, for an arbitrary fixed τ, is given by

$$\sum e^{i\lambda_n\tau}\chi_n(S_l)\,a_n\,e^{i\lambda_n t},$$

and as, throughout, we deal only with functions of our majorisable set $\{g(t)\}$, the inequality

$$|f_l(t+\tau)-S_h f_l(t)|\leqq\varepsilon$$

is (by the remark above) certainly fulfilled by any τ for which the N inequalities

$$|e^{i\lambda_n\tau}\chi_n(S_l)\,a_n-\chi_n(S_h)\,\chi_n(S_l)\,a_n|\leqq\delta \quad (n=1,\ldots,N),$$

i. e. (since $|\chi_n(S_l)|=1$) the N inequalities

$$(15)\qquad \left|\left(e^{i\lambda_n\tau}-\chi_n(S_h)\right)a_n\right|\leqq\delta \quad (n=1,\ldots,N),$$

are fulfilled. We see that the index l has disappeared, so that any τ satisfying the inequalities (15) will certainly satisfy the m inequalities (14). Denoting by a the maximum of $|a_1|,\ldots,|a_N|$, the inequalities (15) are certainly satisfied by any τ which satisfies

$$(16)\qquad |e^{i\lambda_n\tau}-\chi_n(S_h)|\leqq\frac{\delta}{a} \quad (n=1,\ldots,N).$$

But according to the "non-trivial" part of KRONECKER's theorem, the inequalities (16) certainly have a solution τ since condition 4 assures us that to every linear relation with integral coefficients such as $g_1\lambda_1+\ldots+g_N\lambda_N=0$ there corresponds the relation $\left(\chi_1(S_h)\right)^{g_1}\ldots\left(\chi_N(S_h)\right)^{g_N}=1$. Thus the proof of theorem 6 is completed.

We can now easily prove the following corollary, which is to be considered one of the main results of the paper.

Corollary. *Corresponding to each arbitrarily given transitive Abelian group Γ of substitutions on the indices $1, \ldots, m$, there exists a set $[f(t)]$ of m distinct almost-periodic functions which has Γ as its almost-translation group.*

Proof. We have only to show that to the given group Γ there correspond m almost-periodic functions $f_1(t), \ldots, f_m(t)$ whose Fourier series satisfy conditions 1—4 of theorem 6. In order to avoid any trouble arising from the somewhat intricate condition 4, we choose the exponents λ_n rationally independent, i. e. such that a relation $g_1\lambda_1 + \ldots + g_N\lambda_N = 0$ with integral coefficients can occur only if every g is zero. Condition 4 then falls away. Next, to be sure that the series we set up are the Fourier series of almost-periodic functions, we limit ourselves to only a finite number of terms. Now let $\chi'_1, \ldots, \chi'_q$ be any generating system of the character-group Γ^*, let $\lambda_1, \ldots, \lambda_q$ be arbitrarily chosen rationally independent real numbers, and let $a_1, \ldots, a_q$ be arbitrary complex numbers $\neq 0$. Then it is clear that the m almost-periodic functions

$$f_h(t) = \sum_{n=1}^{q} \chi'_n(S_h)\, a_n e^{i\lambda_n t} \qquad (h = 1, \ldots, m)$$

satisfy conditions 1—4, and so the set $[f(t)]$ has Γ as its almost-translation group.

If we wish to have as few terms as possible in the example just constructed we shall take $q = \mu$, where μ is the number of elements in a minimal generating system of Γ (or of Γ^*), mentioned in section I; and for our characters $\chi'_1, \ldots, \chi'_\mu$ we shall take any minimal generating system of Γ^*. We shall show that the example thus obtained of a set $[f(t)]$ of functions belonging to the given group Γ is not only a simplest possible example, in that the Fourier series contains as few terms as possible, but at the same time is the most general example of a set $[f(t)]$ belonging to Γ, for which the functions $f_h(t)$ contain the minimum number μ of terms. By this we mean that the only restriction (other than as to the number of terms) of a voluntary character which has been introduced so far, namely *that the exponents be rationally independent*, is in fact a necessary one. We shall prove, namely, the following general theorem:

Theorem 7. Let

$$f_h(t) \sim \sum \chi_n(S_h) a_n e^{i\lambda_n t} \quad (h = 1, \ldots, m)$$

be any m almost-periodic functions with the transitive group Γ as almost-translation group of the set $[f(t)]$, and let μ denote the number of elements in a minimal generating system of Γ. Then among the exponents $\lambda_1, \lambda_2, \ldots$ there occur at least μ which are rationally independent.

Proof. Among the characters χ_n involved in the Fourier series there certainly occurs some generating system, $\chi'_1, \ldots, \chi'_q$, of Γ^*. By a remark in section I, out of this generating system there can always be chosen a set of just μ characters, say $\varphi_1, \ldots, \varphi_\mu$, such that any relation $\varphi_1^{g_1} \ldots \varphi_\mu^{g_\mu} = 1$ implies $GCD(g_1, \ldots, g_\mu) > 1$. Now let $n_1, \ldots, n_\mu$ be arbitrarily chosen indices such that

$$\chi_{n_1} = \varphi_1, \ldots, \chi_{n_\mu} = \varphi_\mu.$$

Then the exponents $\lambda_{n_1}, \ldots, \lambda_{n_\mu}$ must be rationally independent; otherwise there would exist a linear relation

$$g_1 \lambda_{n_1} + \cdots + g_\mu \lambda_{n_\mu} = 0$$

with $GCD(g_1, \ldots, g_\mu) = 1$. But by condition 4 this would imply

$$\chi_{n_1}^{g_1} \cdots \chi_{n_\mu}^{g_\mu} = 1, \text{ i. e. } \varphi_1^{g_1} \cdots \varphi_\mu^{g_\mu} = 1,$$

contrary to our hypothesis that no such relation can exist with $GCD(g_1, \ldots, g_\mu) = 1$.

In the case considered, where the almost-translation group Γ is transitive, the characters of Γ enable us to give a certain canonical representation of the m functions in the set $[f(t)]$, in which all the functions $f_h(t)$ are represented as linear combinations of a finite set of almost-periodic functions which do not depend on h.

Starting from the particular form of the Fourier series of the functions $f_h(t)$ given in condition 2 of theorem 6, namely

(17) $$f_h(t) \sim \sum \chi_n(S_h)\, a_n e^{i\lambda_n t} \quad (h = 1, \ldots, m),$$

we obtain our representation formally by collecting those terms for which the characters are the same. Thus we may write

(18) $$f_h(t) \sim \sum_{\chi \varepsilon \Gamma^*} \chi(S_h) \left(\sum_{\chi_n = \chi} a_n e^{i\lambda_n t} \right) \quad (h = 1, \ldots, m),$$

where, for the sake of uniformity of notation, we suppose that $\sum\limits_{\chi_n = \chi} a_n e^{i\lambda_n t}$ is simply empty if χ does not occur in the series in (17). Since the series in ( )'s on the right side of the relations (18) are independent of h, the principal step in establishing our canonical representation is to prove that these series are the actual Fourier series of some almost-periodic functions of t.

In general, a subseries of the Fourier series of an almost-periodic function is not itself the Fourier series of any almost-periodic function. In our case, however, each series

$$(19) \qquad \sum_{\chi_n = \chi} a_n e^{i\lambda_n t},$$

is the Fourier series of an almost-periodic function.

For if $\psi(S)$ is any fixed character and we multiply the relations (18) by the respective values $\overline{\psi}(S_1), \ldots, \overline{\psi}(S_m)$ of the conjugate character $\overline{\psi}$ and add the resulting relations, we have

$$\sum_{h=1}^{m} \overline{\psi}(S_h) f_h(t) \sim \sum_{h=1}^{m} \overline{\psi}(S_h) \left\{ \sum_{\chi \varepsilon \Gamma^*} \chi(S_h) \left(\sum_{\chi_n = \chi} a_n e^{i\lambda_n t} \right) \right\}$$

or

$$(20) \quad \sum_{h=1}^{m} \overline{\psi}(S_h) f_h(t) \sim \sum_{\chi \varepsilon \Gamma^*} \left(\sum_{h=1}^{m} \overline{\psi}(S_h) \chi(S_h) \right) \left(\sum_{\chi_n = \chi} a_n e^{i\lambda_n t} \right).$$

Now, due to the orthogonality of the characters, $\sum_{h=1}^{m} \overline{\psi}(S_h) \chi(S_h) = m$ or 0 according as χ is or is not the chosen character ψ. Hence after dividing by m the relation (20) becomes

$$\frac{1}{m} \sum_{h=1}^{m} \overline{\psi}(S_h) f_h(t) \sim \sum_{\chi_n = \psi} a_n e^{i\lambda_n t}.$$

Replacing the letter ψ by χ, we see that the series (19) is the Fourier series of the almost-periodic function

$$\Phi_\chi(t) = \frac{1}{m} \sum_{h=1}^{m} \overline{\chi}(S_h) f_h(t) \sim \sum_{\chi_n = \chi} a_n e^{i\lambda_n t}.$$

Hence, by elementary theorems on almost-periodic functions, the Fourier series of the function

$$\sum_{\chi \varepsilon \Gamma^*} \chi(S_h)\, \Phi_\chi(t)$$

is just the Fourier series of $f_h(t)$, as written in (18), so that

$$f_h(t) = \sum_{\chi \varepsilon \Gamma^*} \chi(S_h)\, \Phi_\chi(t) \quad (h = 1, \ldots, m)$$

which is our desired canonical representation of the functions $f_h(t)$ as linear combinations of a common set of m almost-periodic functions, namely the functions $\Phi_\chi(t)$.

In the next section, where we shall consider functions $f_h(t)$ which are the roots of an algebraic equation with almost-periodic coefficients, we shall return to this canonical representation and prove an interesting property concerning the connection between the Fourier exponents of each function $\Phi_\chi(t)$ and those of the coefficients of the equation.

VII. Algebraic Equations with Almost-Periodic Coefficients whose Roots have a Transitive Almost-Translation Group.

In this last section we return to the consideration of the algebraic equation

$$(21) \quad y^m + x_1(t)\, y^{m-1} + \ldots + x_{m-1}(t)\, y + x_m(t) = 0$$

with almost-periodic coefficients. In case the discriminant $D(t)$ satisfies the condition

$$(22) \qquad \underset{-\infty < t < +\infty}{G\,L\,B} \; |D(t)| > 0,$$

we know that the roots $y_h(t)$ are again almost-periodic functions. We put the following

Problem: To indicate necessary and sufficient conditions which m functions, $y_1(t), \ldots, y_m(t)$, assumed to be almost-periodic, must fulfill in order that

(a) they may be the roots of an algebraic equation (21) of degree m with (eo ipso almost-periodic) coefficients $x_j(t)$ satisfying (22), and

(b) the set $[y(t)]$ may have a given transitive Abelian group Γ as its almost-translation group.

It is not to be expected that an answer to this problem should be as neat as the answer — given in theorem 6 of section VI — to the analogous problem bearing only on the Fourier series of the functions $y_h(t)$. For the condition that the functions be distinct has been replaced by the condition that the discriminant of the equation (21) whose roots they are shall satisfy (22); and it does not seem possible to transform this more complex condition into a simple condition on the Fourier series of the functions. However, for our purpose it suffices to remark that — as the $y_h(t)$'s are almost-periodic, and hence bounded — the demand that the $y_h(t)$'s satisfy (22) is obviously equivalent to requiring that

$$\text{(23)} \qquad \underset{\substack{-\infty < t < +\infty \\ h \neq g}}{G\,L\,B} \left| y_h(t) - y_g(t) \right| > 0.$$

Thus we may give the following

Answer to the Problem: In order that the almost-periodic functions $y_h(t)$ shall satisfy the conditions (a) and (b) in question, it is necessary and sufficient that they satisfy the condition (23), and that their Fourier series satisfy conditions 1—4 of theorem 6.

With the help of this answer we now easily reach the following principal result, already alluded to in the introduction.

Theorem 8. *Corresponding to any arbitrarily given transitive Abelian group Γ, there exists an algebraic equation* (21) *with almost-periodic coefficients $x_j(t)$ satisfying* (22) *and such that the set $[y(t)]$ of its (eo ipso almost-periodic) roots has Γ as its almost-translation group.*

Proof. If as in section VI we denote by $\chi_1', \ldots, \chi_\mu'$ an arbitrarily chosen minimal generating system of the character group Γ^*, by $a_1, \ldots, a_\mu$ arbitrary complex numbers (each $\neq 0$), and by $\lambda_1, \ldots, \lambda_\mu$ arbitrary rationally independent real numbers, we know already that the m (distinct) almost-periodic functions

$$y_h(t) = \sum_{n=1}^{\mu} \chi_n'(S_h)\, a_n e^{i\lambda_n t} \qquad (h = 1, \ldots, m)$$

satisfy the conditions 1—4 of theorem 6. We complete the proof by showing that we can easily make these functions satisfy the additional condition (23), simply by choosing the (thus far arbitrary) coefficients a_n so that the sequence $|a_1|, \ldots, |a_\mu|$ of their absolute values decreases rather strongly. In order not to spoil too much the generality of our example by putting too great restrictions on the $|a_n|$'s, we base our limitation on the respective orders γ_n $(1 < \gamma_n \leq m)$ of the characters χ_n' in our minimal generating system $\chi_1', \ldots, \chi_\mu'$. Since $(\chi_n')^{\gamma_n} = 1$, so that each of the m values $\chi_n'(S_1), \ldots, \chi_n'(S_m)$ is a γ_n-th root of unity, for any two distinct substitutions S_h and S_g of Γ the difference

$\chi_n'(S_h) - \chi_n'(S_g)$ is either 0 or numerically $\geq 2 \sin \frac{\pi}{\gamma_n}$. Hence for any two distinct functions $y_h(t)$ and $y_g(t)$ and any real t it is obvious that

$$|y_h(t) - y_g(t)| = \left| \sum_{n=1}^{\mu} a_n \left(\chi_n'(S_h) - \chi_n'(S_g)\right) e^{i\lambda_n t} \right| \geq$$

$$2\,|a_p| \sin \frac{\pi}{\gamma_p} - 2\,(|a_{p+1}| + \ldots + |a_\mu|),$$

where $p = p(h, g)$ denotes the smallest (certainly existing) index among $1, \ldots, \mu$ for which $\chi_p'(S_h) \neq \chi_p'(S_g)$. Thus we see that our condition (23) is certainly satisfied if we choose the coefficients $a_1, \ldots, a_\mu$ such that for each $n = 1, \ldots, \mu$,

$$(24) \qquad A_n = |a_n| \sin \frac{\pi}{\gamma_n} - (|a_{n+1}| + \ldots + |a_\mu|) > 0;$$

in fact these conditions imply

$$\underset{\substack{-\infty < t < +\infty \\ h \neq g}}{G\,L\,B} |y_h(t) - y_g(t)| \geq 2 \cdot \min(A_1, \ldots, A_\mu) > 0.$$

Remark. If we wish to have conditions on the a_n's which depend only on the group Γ rather than on the choice of the minimal generating system $\chi_1', \ldots, \chi_\mu'$ of Γ^*, we may replace the conditions (24) by the somewhat stronger conditions

$$(25) \quad |a_n| \sin \frac{\pi}{\gamma} - (|a_{n+1}| + \ldots + |a_\mu|) > 0 \quad (n = 1, \ldots, \mu),$$

where γ ($\geq$ each γ_n) denotes the highest order of any element χ in Γ^* (or, equivalently, of any element S in Γ); or we may go further and replace γ by m.

Example. As the simplest possible example of a transitive but not cyclic Abelian substitution group, we consider the group Γ of the following four substitutions:

$$S_1 = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,4 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,1\,4\,3 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix} \qquad S_4 = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix}$$

for which a complete table of character values is

	χ_1	χ_2	χ_3	χ_4
S_1	1	1	1	1
S_2	1	1	-1	-1
S_3	1	-1	1	-1
S_4	1	-1	-1	1.

Here $\mu = 2$, and a minimal generating system of Γ^* is formed by any pair from χ_2, χ_3, χ_4, say by χ_2, χ_3. Hence, if we let λ_1 and λ_2 be any two real numbers with an irrational ratio, and let a_1 and a_2 have any complex values other than 0, the four functions

$$\begin{aligned} y_1(t) &= \quad a_1 e^{i\lambda_1 t} + a_2 e^{i\lambda_2 t} \\ y_2(t) &= \quad a_1 e^{i\lambda_1 t} - a_2 e^{i\lambda_2 t} \\ y_3(t) &= -a_1 e^{i\lambda_1 t} + a_2 e^{i\lambda_2 t} \\ y_4(t) &= -a_1 e^{i\lambda_1 t} - a_2 e^{i\lambda_2 t} \end{aligned}$$

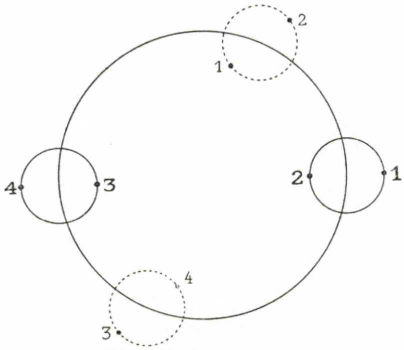
will satisfy conditions 1—4 of theorem 6, and hence have Γ as almost-translation group; and will satisfy the algebraic equation

$$y^4 - 2y^2\left(a_1^2 e^{2i\lambda_1 t} + a_2^2 e^{2i\lambda_2 t}\right) + \left(a_1^2 e^{2i\lambda_1 t} - a_2^2 e^{2i\lambda_2 t}\right)^2 = 0$$

of type (21). If in addition (since in (25) we have $\gamma = \gamma_1 = \gamma_2 = 2$) we take $|a_1| > |a_2| > 0$, the functions will satisfy (23), that is the discriminant of the equation will satisfy (22).

In the figure, where we have chosen $a_1 > a_2 > 0$, the positions in the complex plane of the four functions $y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$ are denoted respectively by **1**, **2**, **3**, **4** when $t = 0$, and by 1, 2, 3, 4 for some other value of t.

In order to show that this set of four almost-periodic functions actually has the non-cyclic group Γ as its almost-translation group — which simple observation was a starting-point of this investigation — we need not, of course, refer to the general theory above. In fact, it is obvious from the very figure that the almost-translation group of the set can contain no other substitutions than those in Γ. That each of these four substi-



tutions actually is an almost-translation substitution of this set of functions, can be verified immediately with the help of KRONECKER's theorem for the case $N = 2$.

Remark 1. In the special, classical, case where the coefficients $x_j(t)$ of the equation (21) are continuous pure periodic functions with a common period, say with least positive common period p, (and where the condition $GLB\,|D(t)| > 0$ is equivalent to the condition $D(t) \neq 0$ for $0 \leqq t < p$), the case lies so clear and is so well known that we may leave to the reader the slight task of showing how the theory developed in this paper for the general case of almost-periodic coefficients may be applied to deriving the periodicity of the roots, and the essential features of their (ordinary) Fourier series. There is one point however which it may be worth while to emphasize: namely, that in this case the almost-translation group Γ (whether it is transitive or intransitive) becomes, as is to be expected, merely the usual "exact translation group" of the set of roots, and hence is simply the cyclic group composed of all powers of that substitution S which is (exactly) performed by p. For it is clear on the one hand, that any integral multiple gp of p exactly performs S^g i. e. ε-performs S^g for every positive ε; and on the other, that each "fine" translation number of the majorant $X(t)$ must be "very near" some gp, and hence must ε-perform the same substitution S^g which gp exactly performs.

Remark 2. Of somewhat greater interest is the case where the coefficients $x_j(t)$ in equation (21) are limit-periodic functions with a common period p; that is, expressed in terms of the exponents, the case in which the module M_X contains only rational multiples of a single number, namely $\alpha = \frac{2\pi}{p}$. From the general relation

$$M_Y \subseteq \frac{1}{m!} M_X$$

it follows that the exponents of the roots $y_h(t)$ likewise are rational multiples of α, and hence *the roots themselves are limit-periodic functions with common period p*. We may also remark that the really interesting case of a non-cyclic transitive group — interesting because it can never occur when the roots are pure periodic — is equally impossible in the limit-periodic case. That is to say, in the limit-periodic case *a transitive almost-translation group Γ must be a cyclic group*, i. e. the number μ must be equal to 1. For, by theorem 7 of section VI, the assumption $\mu > 1$ would imply that the module M_Y contained at least two rationally independent numbers, contrary to what we have just seen.

Finally we prove the following theorem concerning the canonical representation of the functions $y_h(t)$ mentioned at the end of section VI. We suppose here that $[y(t)]$ is the set of roots of an algebraic equation (21) with almost-periodic coefficients $x_j(t)$ satisfying condition (22), and that its almost-translation group Γ is transitive.

We denote the canonical representation of the functions $y_h(t)$ by

$$y_h(t) = \sum_{\chi \varepsilon \Gamma^*} \chi(S_h)\, \Phi_\chi(t) \quad (h = 1, \ldots, m), \tag{26}$$

where the almost-periodic functions

$$\Phi_\chi(t) \sim \sum_{\chi_n = \chi} a_n e^{i\lambda_n t}$$

are independent of h. As before, $X(t)$ denotes a majorant of the $x_j(t)$'s, and M_X the module of their Fourier exponents.

Theorem 9. *If χ is any fixed character in Γ^*, then the difference $\lambda' - \lambda''$ of any two Fourier exponents of $\Phi_\chi(t)$ is contained in the module M_X.*

Proof. Let λ' and λ'' be two distinct Fourier exponents of the function $\Phi_\chi(t)$. (The proof either "goes by default" or is trifling in case $\Phi_\chi(t)$ is identically 0, or has just one Fourier exponent). Let $a'e^{i\lambda' t}$ and $a''e^{i\lambda'' t}$ be the corresponding terms in the Fourier series of $\Phi_\chi(t)$ (and hence also in the Fourier series of $f_1(t)$).

In order to prove that $\lambda' - \lambda''$ belongs to M_X it suffices, from lemma 2 in section IV, to prove that to any $\varepsilon > 0$ there corresponds a $\delta = \delta(\varepsilon) > 0$ such that the inequality

$$|e^{i(\lambda' - \lambda'')\tau} - 1| \leqq \varepsilon,$$

i. e. the inequality

$$(27) \qquad \left| e^{i\lambda'\tau} - e^{i\lambda''\tau} \right| \leq \varepsilon,$$

is satisfied by every τ belonging to $\{\tau_X(\delta)\}$. Now, since

$$\{\tau_X\} \subseteq \{\tau_{[y]}\} \quad \text{and} \quad \{\tau_{[y]}\} \subset \{\tau_X\},$$

this is equivalent to proving that to the given ε there corresponds a $\delta' = \delta'(\varepsilon) > 0$ such that (27) is fulfilled by every τ belonging to $\{\tau_{[y]}(\delta')\}$. But for δ sufficiently small (in fact for $\delta \leq \varepsilon^{**}$) we know from remark 4° in section II that $\{\tau_{[y]}(\delta')\}$ is equal to the set-theoretical sum of the m sets $\{\tau_{(S)}(\delta')\}$, where S belongs to Γ. Hence it suffices to prove that to the given ε there corresponds a $\delta'' = \delta''(\varepsilon) > 0$ such that the inequality (27) is satisfied whenever τ belongs to one of the sets $\{\tau_{(S_h)}(\delta'')\}$. We complete the proof by showing that $\frac{\varepsilon}{2}a$, where a denotes $\min(|a'|, |a''|)$, is a suitable value of δ''.

In fact, let τ be a fixed number belonging to any one of the m sets in question, say to $\{\tau_{(S_h)}(\delta'')\}$. Then by the very definition of the translation numbers $\tau_{(S_h)}(\delta'')$ we have

$$(28) \quad |y_1(t+\tau) - y_h(t)| \leq \delta'' = \frac{\varepsilon}{2}a \quad (\text{for } -\infty < t < +\infty).$$

In the Fourier series of the almost-periodic function

$$y_1(t+\tau) - y_h(t)$$

the coefficients belonging to the exponents λ' and λ'' are given by

$$a' e^{i\lambda'\tau} - a'\chi(S_h) \quad \text{and} \quad a'' e^{i\lambda''\tau} - a''\chi(S_h)$$

respectively. As any Fourier coefficient of an almost-periodic function $f(t)$ is numerically $\leq L\,U\,B\,|f(t)|$, we conclude from (28) that

$$|a' e^{i\lambda'\tau} - a'\chi(S_h)| \leqq \frac{\varepsilon}{2} a \quad \text{and} \quad |a'' e^{i\lambda''\tau} - a''\chi(S_h)| \leqq \frac{\varepsilon}{2} a,$$

and hence (dividing by $|a'|$ and $|a''|$ respectively)

$$|e^{i\lambda'\tau} - \chi(S_h)| \leqq \frac{\varepsilon}{2} \quad \text{and} \quad |e^{i\lambda''\tau} - \chi(S_h)| \leqq \frac{\varepsilon}{2}.$$

Thus τ satisfies the inequality $|e^{i\lambda'\tau} - e^{i\lambda''\tau}| \leqq \varepsilon$, i. e. the inequality (27), and the proof is complete.

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